

On Some Closed Sets In Fuzzy Minimal Ideal Spaces

Anita Singh Banafar¹, Sudhir Prakash², Anil Kumar³

¹Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur 482011

² Department of Applied Science, IET, Dr.RML Avadh University, Ayodhya, 224001

³Department of Mathematics, School of Physical Science, Starex University Gurugram

Abstract: The purpose of this paper is to introduce fuzzy minimal ideal topological spaces and fuzzy minimal ideal generalized closed sets in fuzzy minimal topological spaces and to investigate the relationships between fuzzy minimal spaces and fuzzy minimal ideal spaces.

Keywords: Fuzzy ideal topological spaces, Fuzzy I_g - closed sets, Fuzzy minimal ideal space, Fuzzy minimal ideal generalized closed sets etc.

Introduction

In 1945, R. Vaidyanathaswamy[7] introduced the concept of ideal topological spaces. Hayashi[4] defined the local function and studied some topological properties using local function in ideal topological spaces in 1964. Since then many mathematicians studied various topological concepts in ideal topological spaces. After the introduction of fuzzy sets by Zadeh[5] in 1965 and fuzzy topology by Chang[2] in 1968, several mathematicians[10, 3, 6] studied the generalization of fuzzy sets and fuzzy ideal topological spaces. In this sequence Noiri and Popa[11] have created and developed the theory of minimal structure in general topology. As a generalization of the fuzzy domain, Brescan[6] introduced the concept of fuzzy minimal structure. This paper explores the fuzzy minimal ideal topological spaces and its generalization in fuzzy ideal topological spaces.

Preliminary

Let X be a nonempty set. A family r of fuzzy sets of X is called a fuzzy topology on X if the null fuzzy set 0 and the whole fuzzy set 1 belongs to r and r is closed with respect to any union and finite intersection[5]. If r is a fuzzy topology on X , then the pair (X, r) is called a fuzzy topological space [5] and the members of r are called fuzzy open sets and their complements are called fuzzy closed sets. The closure of a fuzzy set A of X denoted by (A) , is the intersection of all fuzzy closed sets which contains A . The interior of a fuzzy set A of X denoted by $Ikt(A)$ is the union of all fuzzy sets of X contained in A . A fuzzy set A in fuzzy topological space (X, r) is said to be quasi-coincident with a fuzzy set B denoted by AqB , if there exists a point $x \in X$ such that $A(x) + B(x) > 1$ [9]. A fuzzy set V in a fuzzy topological space (X, r) is called a Q -neighborhood of a fuzzy point x_β if there exists a fuzzy open set U of X such that $x_\beta qU \leq V$ [3].

Definition 2.1: A nonempty collection of fuzzy sets I of a set X satisfying the conditions

(i) if $A \in I$ and $B \leq A$, then $B \in I$ (heredity),

(ii) if $A \in I$ and $B \in I$ then $A \cup B \in I$ (finite additivity) is called a fuzzy ideal on X . The triplex (X, r, I) denotes a fuzzy ideal topological space with a fuzzy ideal I and fuzzy topology r [5, 9].

Definition 2.2: The local function for a fuzzy set A of X with respect to r and I denoted by $A^*(r, I)$ (briefly A^*) in a fuzzy ideal topological space (X, r, I) is the union of all fuzzy points x_β such that if U is a Q -neighbourhood of x_β and $E \in I$ then for at least one point $y \in X$ for which $U(y) + A(y) - 1 > E(y)$ [9].

Definition 2.3: The Closure of a fuzzy set A denoted by $Cl^*(A)$ in (X, r, I) defined as $Cl^*(A) = A \cup A^*$ [9].

In a fuzzy ideal topological space (X, r, I) , the collection $r^*(I)$ means an extension of fuzzy topological space than r via fuzzy ideal which is constructed by considering the Class $\beta = \{U - E : U \in r, E \in I\}$ as a base [9].

Definition 2.4: A fuzzy set A of a fuzzy ideal topological space (X, r, I) is called fuzzy I_g -closed if $A^* \leq U$, whenever $A \leq U$ and U is fuzzy open. And A is called called fuzzy I_g -open if its complement is fuzzy I_g -closed [10].

Remark 2.5: Every fuzzy g -closed (resp. fuzzy $*$ -closed) set is fuzzy I_g -closed and every fuzzy g -open (resp. fuzzy $*$ -open) set is fuzzy I_g -open. Examples can be easily constructed to show that the converse may not be true [10].

Remark 2.6: In a fuzzy ideal topological space (X, r, I) , I is fuzzy I_g -closed for every $I \in I$ [10].

Lemma 2.7: $A \leq B \Leftrightarrow \bigcap (Aq1 - B)$, for every pair of fuzzy sets A and B of X [11].

Definition 2.8: Let $F(X)$ be a class of fuzzy subsets of a non empty set X . A subclass F_m of $F(X)$ is called fuzzy minimal structure on X (or a F_m -structure) if $0 \in F_m, 1 \in F_m$. The pair (X, F_m) is called fuzzy minimal space or a F_m -space. A fuzzy set A is called F_m -open set if $A \in F_m$, if $A^c \in F_m$ then A is called F_m -closed set[6].

Remark 2.9: Fuzzy minimal structure on X maintains only the first condition of the definition of fuzzy topology[6].

Remark 2.10: Let (X, r) be a fuzzy topological space. Then the subfamilies $r, FSO(X), FPO(X), F\delta PO(X), FSPO(X), FS\delta PO(X)$, of $F(X)$ are all fuzzy minimal structures on X [6].

Definition 2.11: Let X be a nonempty set and F_m be a fuzzy minimal structure on X . Then the F_m -closure and the F_m -interior of a fuzzy set of X are defined in as follows[6]:

(i) $F_m Cl(A) = \bigcap \{\sigma : A \subseteq \sigma, \sigma^c \in F_m\}$,

(ii) $F_m Ikt(A) = \bigcup \{\delta : \delta \subset A, \delta \in F_m\}$.

Lemma 2.12: Let (X, F_m) be a F_m -space on X and $A \in F_m$ and x_α be a fuzzy point in X . Then $x_\alpha \in F_m Cl(A)$ if and only if μqA for any fuzzy set $\mu \in F_m$ satisfying the condition $x_\alpha q\mu$. [6].

3.1 Fuzzy Minimal Ideal space

Definition 3.1: Let $F(X)$ be a class of fuzzy subsets of a non empty set X . A subclass F_m of $F(X)$ is called fuzzy minimal ideal structure on X (or a $F_m I$ -structure) (i) if $A \in I$ and $B \leq A$, then $B \in I$

(heredity) , (ii) if $A \in I$ and $B \in I$ then $A \cup B \in I$ (finite additivity) is called a fuzzy ideal on X . The triple (X, F_m, I) denotes a fuzzy minimal ideal topological space .

The local function for a fuzzy set A of X with respect to F_m and I denoted by $A^* (F_m, I)$ (briefly A^*) in a fuzzy minimal ideal topological space (X, F_m, I) is the union of all fuzzy points x_β such that if U is a fuzzy Q-neighbourhood of x_β and $E \in I$ then for at least one point $y \in X$ for which $U(y) + A(y) - 1 > E(y)$.

Remark 3.2: Let (X, F_m, I) be a fuzzy minimal structure. Then

- (i) If $I = \{\emptyset\}$, then $A^*_m(\emptyset) = mCl(A)$,
- (ii) If $I = \wp(X)$, then $A^*_m(\wp(X)) = \emptyset$

Definition 3.3: Let (X, F_m, I) be a fuzzy minimal space with an ideal I on X . For $A \subset X$ the set operator mCl^* is called a fuzzy minimal *-closure ideal and is defined as $mCl^*(A) = A \cup A^*_m$. It is denoted as $m^*_S(X, F_m, I)$ the fuzzy minimal structure generated by mCl^* , that is, $m^*_S(X, F_m, I) = \{U \subset X : mCl^*(X \setminus U) = X - U\}$; $m^*_S(X, F_m, I)$ is called fuzzy *-minimal ideal structure which is finer than F_m .

Remark 3.4:The elements of $m^*_S(X, F_m, I)$ are called fuzzy minimal *-open (briefly, fuzzy m^* -open) and the complement of a fuzzy m^* -open set is called fuzzy minimal *-closed (briefly, fuzzy m^* -closed).

Theorem 3.5: Let (X, F_m) be a fuzzy minimal-space with an fuzzy ideal I on X and A, B be fuzzy subsets of X . Then the following properties hold:

- (i) $A \subset B \Rightarrow A^*_m \subset B^*_m$,
- (ii) $I \subset I' \Rightarrow A^*_m(I) \subset A^*_m(I')$,
- (iii) $A^*_m = mCl(A^*_m) \subset mCl(A)$,
- (iv) $A^*_m \cup B^*_m \subset (A \cup B)^*_m$,
- (v) $(A^*_m)^*_m \subset A^*_m$.

Proof: (i) Since $A \subset B \Rightarrow A^*_m \subset B^*_m$, for every $x \in A^*_m$. Then by the definition $x \in A^*_m$ implies $x \in B^*_m$. Which completes the proof.

(ii) The proof is obvious

(iii) The proof is obvious

(iv) Let $x_\beta \notin A^*_m \cup B^*_m$ therefore x_β is not contained in both A^*_m and B^*_m . Then there is one Q-neighbourhood u_1 of x_β such that for every $y \in X$, $u_1 + A(y) - 1 \leq E_1$, for some $E_1 \in I$. Similarly there is one Q-neighborhood of u_2 of x_β such that for every $y \in X$, $u_2 + A(y) - 1 \leq E_2$, for some $E_2 \in I$. Let $u = u_1 \cup u_2$, u is also Q-neighbourhood of x_β such that $u(y) + A \cup B(y) - 1 \leq E_1 \cup E_2$ for every $y \in X$. Therefore by finite additivity of fuzzy ideal $E_1 \cup E_2 \in I$ and $x_\beta \notin (A \cup B)^*_m$. Hence $A^*_m \cup B^*_m \subset (A \cup B)^*_m$.

(v) The proof is obvious.

Remark 3.6: Let (X, F_m, I) be a fuzzy minimal ideal space with an ideal I on X , then $A^*_m \cup B^*_m = (A \cup B)^*_m$.

Corollary 3.7: Let (X, F_m, I) be a fuzzy minimal space and $A \subset X$. Then the set operator fuzzy $mCl^*(A)$ satisfies the following conditions:

- (i) $A \subset mCl^*(A)$,
- (ii) $mCl(\emptyset) = \emptyset$ and $mCl^*(X) = X$,
- (iii) If $A \subset B$, then $mCl^*(A) \subset mCl^*(B)$,
- (iv) $mCl^*(A) \cup mCl^*(B) \subset mCl^*(A \cup B)$.

4.1 Fuzzy minimal I_g -closed

Definition 4.1: A subset A of fuzzy minimal ideal space (X, F_m, I) is fuzzy minimal- I -generalized closed (in brief, fuzzy minimal I_g -closed) if $A_m^* \subset U$ whenever U is fuzzy minimal-open and denoted as (X, F_m, I_g) .

Proposition 4.2: Let (X, m_S) be a fuzzy minimal space. Every fuzzy minimal-closed set is fuzzy minimal I_g -closed.

Remark 4.3: Every fuzzy minimal *-closed set is fuzzy minimal I_g -closed. But the converse may not be true. Examples can be easily constructed to show that the converse may not be true.

Example 4.4 : Let $X = \{a, b\}$ and A and U be defined as follows:

$$A(a)=0.3, A(b)=0.2$$

$$U(a)=0.5, U(b)=0.7$$

Let $\tau = \{0, U, 1\}$ be the fuzzy minimal topology on X and $I = \{0\}$ be a fuzzy minimal ideal on X . Then A is fuzzy minimal I_g -closed in (X, F_m, I) , but not fuzzy minimal *-closed.

Remark 4.5: Let (X, F_m, I) be a fuzzy minimal ideal space then every fuzzy g -closed set (resp. fuzzy *-closed) set is fuzzy minimal I_g -closed.

Theorem 4.7: Let (X, F_m, I) be a fuzzy minimal ideal topological space. Then A^* is fuzzy minimal I_g -closed for every fuzzy set A of X .

Proof: Let A be a fuzzy set of X and U be any fuzzy minimal open set of X such that $A^* \leq U$. Since $(A^*)^* \leq A^*$ it follows that $(A^*)^* \leq U$. Hence A^* is fuzzy minimal I_g -closed.

Theorem 4.8: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be a fuzzy minimal I_g -closed and fuzzy open set in X . Then A is fuzzy minimal *-closed.

Proof: Since A is fuzzy open and fuzzy minimal I_g -closed and $A \leq A$. It follows that $A^* \leq A$ because A is fuzzy minimal I_g -closed. Hence $Cl^*(A) = A \cup A^* \leq A$ and A is fuzzy minimal *-closed.

Theorem 4.9: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be a fuzzy minimal set of X . Then the following are equivalent:

- (i) A is fuzzy minimal I_g -closed.
- (ii) $Cl^*(A) \leq U$ whenever $A \leq U$ and U is fuzzy minimal open in X .
- (iii) $\bigcap (AqF) \Rightarrow \bigcap (Cl^*(A)qF)$ for every fuzzy minimal closed set F of X .
- (iv) $\bigcap (AqF) \Rightarrow \bigcap (A^*qF)$ for every fuzzy minimal closed set F of X .

Proof: (i) \Rightarrow (ii). Let A be a fuzzy minimal I_g -closed set in X . Let $A \leq U$ where U is fuzzy minimal open set in X . Then $A^* \leq U$. Hence $Cl^*(A) = A \cup A^* \leq U$. Which implies that $Cl^*(A) \leq U$.

(ii) \Rightarrow (i). Let A be a fuzzy set of X . By hypothesis $Cl^*(A) \leq U$. Which implies that $A^* \leq U$. Hence A is fuzzy minimal I_g -closed.

(ii) \Rightarrow (iii). Let F be a fuzzy minimal closed set of X and $\bigcap (AqF)$. Then $1 - F$ is fuzzy minimal open in X and by Lemma 2.7, $\leq 1 - F$. Therefore, $Cl^*(A) \leq 1 - F$, because A is fuzzy minimal I_g -closed. Hence by Lemma 2.7, $\bigcap (Cl^*(A)qF)$.

(iii) \Rightarrow (ii). Let U be a fuzzy minimal open set of X such that $A \leq U$. Then by Lemma 2.7, $\bigcap (Aq(1 - U))$ and $1 - U$ is fuzzy minimal closed in X . Therefore by hypothesis $\bigcap (Cl^*(A)q(1 - U))$. Hence, $Cl^*(A) \leq U$.

(i) \Rightarrow (iv). Let F be a fuzzy minimal closed set in X such that $\bigcap (AqF)$. Then $A \leq 1 - F$ where $1 - F$ is fuzzy minimal open. Therefore by (i) $A^* \leq 1 - F$. Hence $\bigcap (A^*qF)$.

(iv) \Rightarrow (i). Let U be a fuzzy minimal closed set in X such that $A \leq U$. Then by Lemma 2.7, $\bigcap (Aq(1 - U))$ and $1 - U$ is fuzzy minimal closed in X . Therefore by hypothesis $\bigcap (A^*q(1 - U))$. Hence $A^* \leq U$ and A is fuzzy minimal I_g -closed set in X .

Theorem 4.10: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be a fuzzy minimal I_g -closed set. Then $x qCl^*(A) \Rightarrow Cl(x) qA$ for any fuzzy point x of X .

Proof: Let $x qCl^*(A)$. If $\bigcap (Cl(x) qA)$. Then by Lemma 2.7, $A \leq (1 - Cl(x))$. And so by Theorem 3.3(ii), $Cl^*(A) \leq (1 - Cl(x))$ because $(1 - Cl(x))$ is fuzzy minimal open set in X . Which implies that $Cl^*(A) \leq (1 - x)$. Hence by Theorem 3.3(ii), $\bigcap (x qCl^*(A))$, which is a contradiction.

Theorem 4.11: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be a fuzzy minimal I_g -closed set in X . Then A is fuzzy minimal $*$ -closed.

Proof: The proof is obvious.

Theorem 4.12: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be fuzzy $*$ -dense in itself fuzzy minimal I_g -closed set of X . Then A is fuzzy g -closed.

Proof: Let U be a fuzzy minimal open set of X such that $A \leq U$. Since A is fuzzy minimal I_g -closed, by Theorem 4.9 (ii), $Cl^*(A) \leq U$. Therefore, $Cl(A) \leq U$, because A is fuzzy $*$ -dense in itself. Hence A is fuzzy g -closed.

Theorem 4.13: Let (X, F_m, I) be a fuzzy minimal ideal topological space where $I = \{0\}$ and A be a fuzzy minimal set of X . Then A is fuzzy minimal I_g -closed if and only if A is fuzzy g -closed.

Proof: Since $I = \{0\}$, $A^* = Cl(A)$ for each subset A of X . Now the result can be easily proved.

Theorem 4.14: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A, B are fuzzy minimal I_g -closed sets of X . Then $A \cup B$ is fuzzy minimal I_g -closed.

Proof: Let U be a fuzzy minimal open set of X such that $A \cup B \leq U$. Then $A \leq U$ and $B \leq U$. Therefore $A^* \leq U$ and $B^* \leq U$ because A and B are fuzzy minimal I_g -closed sets of X . Hence $(A \cup B)^* \leq U$ and $A \cup B$ is fuzzy minimal I_g -closed.

Remark 4.15: The intersection of two fuzzy minimal I_g -closed sets in a fuzzy minimal ideal topological space (X, F_m, I) may not be fuzzy minimal I_g -closed.

Example 4.16 : Let $X = \{a, b\}$ and A, B be two fuzzy minimal sets defined as follows :

$$\begin{aligned} A(a) &= 0.9 & , & & A(b) &= 0.7 \\ B(a) &= 0.8 & , & & B(b) &= 0.7 \\ U(a) &= 0.3 & , & & U(b) &= 0.4 \end{aligned}$$

Let $\tau = \{0, U, 1\}$ and $I = \{0\}$. Then A and B are fuzzy minimal I_g -closed sets in (X, F_m, I) but $A \cap B$ is not fuzzy minimal I_g -closed.

Theorem 4.17: Let (X, F_m, I) be a fuzzy ideal topological space and A, B are fuzzy minimal sets of X such that $A \leq B \leq Cl^*(A)$. If A is fuzzy minimal I_g -closed set in X , then B is fuzzy minimal I_g -closed.

Proof: Let U be a fuzzy minimal open set such that $B \leq U$. Since $A \leq B$ we have $A \leq U$. Hence, $Cl^*(A) \leq U$ because A is fuzzy minimal I_g -closed. Now $B \leq Cl^*(A)$ implies that $Cl^*(B) \leq Cl^*(A) \leq U$. Consequently B is fuzzy minimal I_g -closed.

Theorem 4.18: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A, B are fuzzy minimal sets of X such that $A \leq B \leq A^*$. Then A and B are fuzzy I_g -closed.

Proof: The proof is obvious.

Theorem 4.19: Let (X, F_m, I) be a fuzzy minimal ideal topological space. If A and B are fuzzy subsets of X such that $A \leq B \leq A^*$ and A is fuzzy minimal I_g -closed. Then $A^* = B^*$ and B is fuzzy minimal $*$ -open in itself.

Proof: Obvious.

Theorem 4.20: Let (X, F_m, I) be a fuzzy minimal ideal topological space and \mathcal{F} be the family of all fuzzy minimal $*$ -closed sets of X . Then $\tau \subset \mathcal{F}$ if and only if every fuzzy set of X is fuzzy minimal I_g -closed.

Proof: Necessity. Let $\tau \subset \mathcal{F}$ and U be a fuzzy minimal open set in X such that $A \leq U$. Now $U \in \tau \Rightarrow U \in \mathcal{F}$. And so $Cl^*(A) \leq Cl^*(U) = U$ and A is fuzzy minimal I_g -closed set in X .

Sufficiency. Suppose that every fuzzy minimal set of X is fuzzy minimal I_g -closed. Let $U \in \tau$. Since U is fuzzy minimal I_g -closed and $U \leq U$, $Cl^*(U) \leq U$. Hence $Cl^*(U) = U$ and $U \in \mathcal{F}$. Therefore $\tau \subset \mathcal{F}$.

Definition 4.21: A fuzzy set A of a fuzzy minimal ideal topological space (X, F_m, I) is called fuzzy minimal I_g -open if its complement $1 - A$ is fuzzy minimal I_g -closed.

Remark 4.22: Every fuzzy g -open set is fuzzy minimal I_g -open. But the converse may not be true.

Remark 4.23: Every fuzzy $*$ -open set in a fuzzy minimal ideal topological space (X, F_m, I) is fuzzy minimal I_g -open. But the converse may not be true.

Theorem 4.24: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A is fuzzy set of X . Then A is fuzzy minimal I_g -open if and only if $F \leq Ikt^*(A)$ whenever F is fuzzy minimal closed and $F \leq A$.

Proof: Necessity. Let A be fuzzy minimal I_g -open and F is fuzzy minimal closed set such that $F \leq A$. Then $1 - A$ is fuzzy minimal I_g -closed, $1 - A \leq 1 - F$ and $1 - F$ is fuzzy minimal open in X . Hence $Cl^*(1 - A) \leq (1 - F)$. Which implies that $F \leq Ikt^*(A)$.

Sufficiency. Let U be a fuzzy minimal open set such that $1 - A \leq U$. Then $1 - U$ is fuzzy minimal closed set of X such that $1 - U \leq A$. And so by hypothesis, $1 - U \leq Ikt^*(A)$. Which implies that $Cl^*(1 - A) \leq U$ and $1 - A$ is fuzzy minimal I_g -closed. Hence A is fuzzy minimal I_g -open.

Theorem 4.25: Let (X, F_m, I) be a fuzzy minimal ideal topological space and A be a fuzzy set of X . If A is fuzzy minimal I_g -open and $Ikt^*(A) \leq B \leq A$, then B is fuzzy minimal I_g -open.

Proof: Follows from Definition 4.21 and Theorem 4.18.

References:

1. A.A. Naseef and Hatir, On fuzzy pre-I-open sets and a decomposition of fuzzy-I-continuity, *Chaos. Solitons and Fractal*, doi:10-1016j(2007).10
2. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* 24(1968) 182-190.1
3. D. Sarkar, Fuzzy ideal theory, Fuzzy local function and generated fuzzy topology, *Fuzzy Sets and Systems*, 87 (1997) 117-123.13
4. E. Hayashi, Topologies defined by local properties, *Math. Ann.* 156 (1964) 114-178.
5. L. A. Zadeh, Fuzzy Sets, *Information Control (Shenyang)*, 8(1965) 338-353.19
6. M. Brescan, Structure Floues Minimaux, *Petroleum-Gas University of Ploiesti Bulletin, Mathematics-Informatics-Physics Series, Volume LIII, No.1(2001) 115-122.*
7. M. K. Gupta and Rajneesh. Fuzzy γ -I- open sets and a new decomposition of fuzzy semi-I-continuity via fuzzy ideals, *Int. Jour. Math. Analysis.* Vol. 3(28) (2009), 1349-1357.3
8. R. Vadyanathaswami, The localization theory in set topology, *Proceedings of the Indian National Science Academy*, 20(1945) 51-61.15
9. R. Vaidyanathaswamy, *Set topology*, Chelsea, New York, (1960).16
10. S. S. Thakur and Anita S. Banafar, Generalized closed sets in fuzzy ideal topological spaces, *J. Fuzzy Math.* 21(4) (2013), 803-808.
11. T. Noiri and V. Popa, On m-continous functions, *Analele University "Dunarea de Jos" Galati, Fascicola II, XVII(2000) 31-41.*